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BEHAVIOR OF A CLOSED END HOLLOW CYLINDER ACTED UPON BY EXTERNAL PRESSURE

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ABSTRACT: The equations for plastic flow in a cylinder subjected to external pressure are derived. The case where reverse yielding occurs is also considered. The derived equations give results that are very similar to those for internal pressure application.

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This report is the result of a continuing effort to provide high strength, high performance guns for launching high velocity aerodynamic models in the ballistics ranges at the U. S. Naval Ordnance Laboratory.

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Captain, USN
Commander

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R. KENNETH LOBB
By direction

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ILLUSTRATIONS

SYMBOLS

a	inner radius of cylinder
b	outer radius of cylinder
c	radius of interface between plastic and elastic regions where pressure is reapplied to the reverse yielded cylinder
d	radius of interface between once yielded outer region and twice yielded inner region refers to fully autofrettaged cylinder when external pressure is released
n	radius to which plastic flow has occurred upon first application of pressure
p	external pressure applied to cylinder
q	interface pressure between elastic and plastic regions
r	radius
s	combined stress
W, ω	wall ratio = b/a
Yo	yield stress
σ	stress

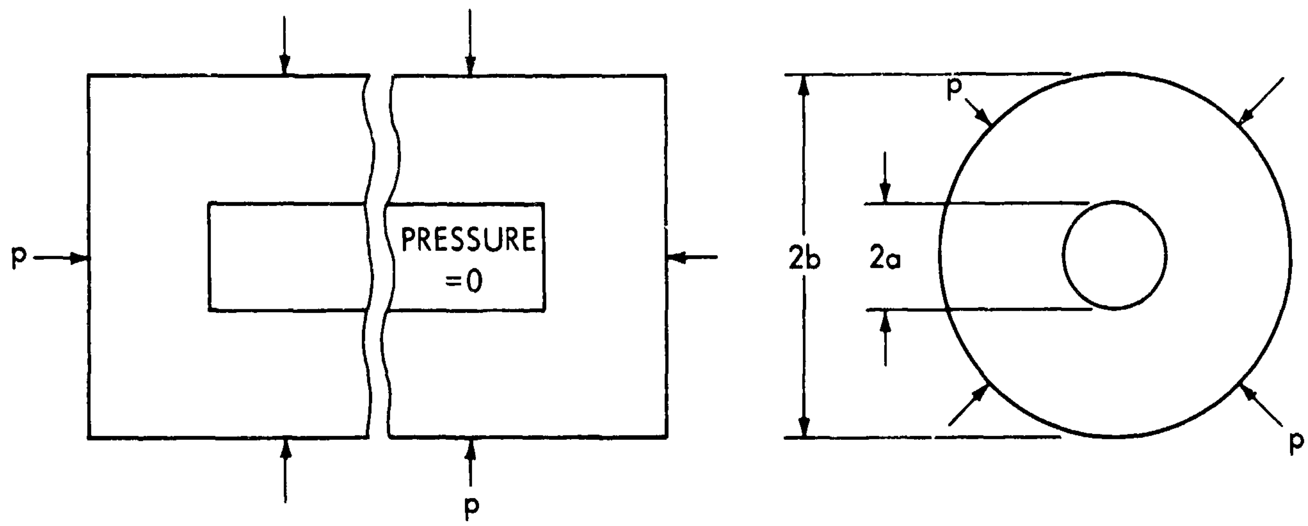
Superscripts

*	residual (stress)

Subscripts

r	radial (stress)
t	tangential (stress)
z	longitudinal (stress)
max	maximum (stress)

A. STRESSES IN CYLINDER DURING PRESSURE APPLICATION BEFORE PLASTIC YIELDING OCCURS



The closed cylinder which is subjected to external pressure is sketched above. The internal radius is denoted by "a"; the external radius by "b". During the application of the external pressure p the stresses in the cylinder before any yielding occurs are given by the following equations (see, for example, references 1, 2, or 3).

$$\sigma_r = -\frac{p}{w^2-1}\left[w^2-\left(\frac{b}{r}\right)^2\right] \tag{1}$$

$$\sigma_t = -\frac{p}{w^2-1}\left[w^2+\left(\frac{b}{r}\right)^2\right] \tag{2}$$

$$\sigma_z = -p\,\frac{w^2}{w^2-1} = \frac{\sigma_r+\sigma_t}{2} \tag{3}$$

where $w \equiv b/a$.

As the value of the external pressure is increased the stresses existing in the cylinder increase. It is assumed that yielding will occur according to the distortion energy theory which states that failure occurs when the combined stress S becomes equal to the value of the yield stress where S is

$$S = \frac{1}{\sqrt{2}} \sqrt{(\sigma_t - \sigma_r)^2 + (\sigma_r - \sigma_z)^2 + (\sigma_z - \sigma_t)^2} \tag{4}$$

From equation (3) for the longitudinal stress, S becomes

$$S = \frac{\sqrt{3}\,(\sigma_t - \sigma_r)}{2} \tag{5}$$

which when equated to the yield stress in compression gives as the yield condition

$$\sigma_t - \sigma_r = \frac{-2Yo}{\sqrt{3}} \tag{6}$$

where Yo is the compressive yield value.

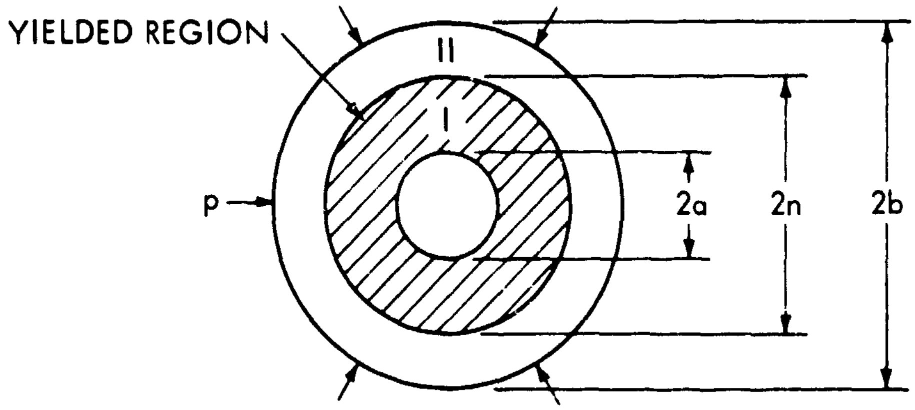
From the elastic equations (1) and (2) for the stresses, it is seen that the value of $\sigma_t - \sigma_r$ is largest at the bore; thus, yielding will first occur at the bore ($r = a$) layer. To determine at what pressure value yielding will first occur at the bore, equations (1) and (2) with $r = a$ are inserted into equation (6). This results in

$$P_{\text{to start yielding at bore}} = \frac{w^2 - 1}{w^2} \; \frac{Yo}{\sqrt{3}} \tag{7}$$

It is interesting that this is the same value of pressure to just cause yielding in the case when only internal pressure is applied (see reference 2, also see section F below).

B. STRESSES IN THE CYLINDER DURING PRESSURE APPLICATION AFTER PLASTIC YIELDING HAS PARTIALLY OCCURRED

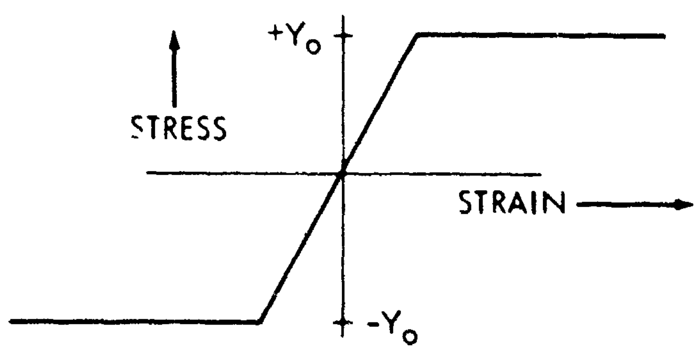
After the external pressure is increased above the value necessary to initiate yielding at the bore (as given by equation (7)), yielding occurs progressively outward from the bore. Let the situation be examined in which the pressure is sufficient to cause plastic yielding to a radius of "n".



In the plastically yielded region I it is again assumed that for the closed end cylinder

$$\sigma_z = (\sigma_r+\sigma_t)/2 \tag{8}$$

(The validity of equation (8) is discussed in reference 2 where it is concluded that it is a justifiable assumption.) The yielded material is assumed to be perfectly plastic, that is, the stress-strain curve appears as sketched below.



Thus, the combined stress S of equation (4) is for the perfectly plastic material equal to the yield strength in compression

$$S = -Yo \tag{9}$$

for the entire yielded region.

Hence, the yielded plasticity condition is the same as the condition for initiation of yield (equation (6)); that is,

$$\sigma_t - \sigma_r = -\frac{2Yo}{\sqrt{3}} \tag{10}$$

describes the entire yielded region.

The equation of equilibrium which is valid throughout the cylinder is

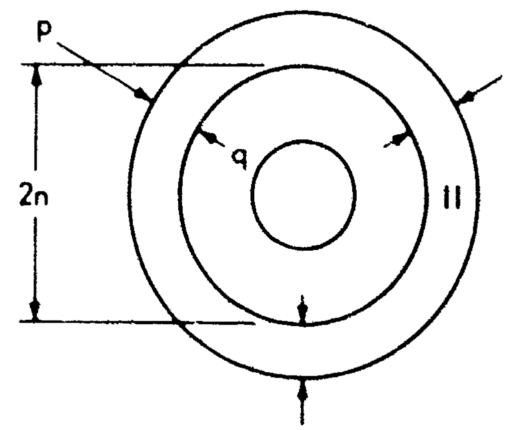
$$\sigma_t - \sigma_r = r\,\frac{d\sigma_r}{dr} \tag{11}$$

From equations (10) and (11)

$$\left.\begin{aligned}\sigma_r &= -\frac{2Yo}{\sqrt{3}}\ln\frac{r}{a}\\ &\text{and}\\ \sigma_t &= -\frac{2Yo}{\sqrt{3}}\left(\ln\frac{r}{a}+1\right)\end{aligned}\right\}\begin{array}{l} n \geq r \geq a\\ \text{(plastic region I)}\end{array} \tag{13}$$

where σ_r has been taken to be zero at $r = a$. Equations (12) and (13) describe the plastically yielded region.

In the outer elastic region (region II) the elastic equations may be written for a cylinder with external pressure p and internal pressure q, where q is the interface pressure between the elastic and plastic regions



$$\sigma_r = \frac{q-(\frac{b}{n})^2 p}{(\frac{b}{n})^2-1} - \frac{(q-p)(\frac{b}{r})^2}{(\frac{b}{n})^2-1} \tag{14}$$

$$\sigma_t = \frac{q-(\frac{b}{n})^2 p}{(\frac{b}{n})^2 -1} + \frac{(q-p)(\frac{b}{r})^2}{(\frac{b}{n})^2 -1} \tag{15}$$

The interface pressure q may be evaluated from the condition that

$$q = -\sigma_r \text{ at } r = n \tag{16}$$

Inserting this condition into equation (12) results in

$$q = \frac{2Yo}{\sqrt{3}} \ln \left(\frac{n}{a}\right) \tag{16}$$

Also at the interface the tangential stress evaluated from the elastic equation must be equal to that evaluated from the plastic equation, or

$$\underset{\substack{\text{elastic}\\ \text{region II}}}{\sigma_t} = \underset{\substack{\text{plastic}\\ \text{region I}}}{\sigma_t} \quad \text{at } r = n \tag{17}$$

Inserting equations(15) and (13) with (16) into equation (17) gives

$$p = \frac{Yo}{\sqrt{3}}\left[2 \ln \frac{n}{a} + 1 - \left(\frac{n}{b}\right)^2\right] \tag{18}$$

the value of external pressure required to yield a cylinder to the radius n. The equations (14) and (15) for the stresses in the elastic region may be rewritten with the above values of p and q inserted.

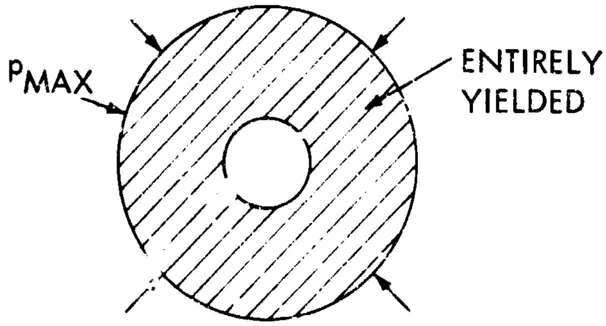
$$\sigma_r = -\frac{Yo}{\sqrt{3}}\left[2 \ln \frac{n}{a} + 1 - \left(\frac{n}{r}\right)^2\right] \qquad \begin{matrix} b \geq r \geq n \\ \text{(elastic region II)} \end{matrix} \tag{19}$$

$$\sigma_t = -\frac{Yo}{\sqrt{3}}\left[2 \ln \frac{n}{a} + 1 + \left(\frac{n}{r}\right)^2\right] \qquad \begin{matrix} b \geq r \geq n \\ \text{(elastic region II)} \end{matrix} \tag{20}$$

In figure 1 is plotted a typical stress distribution in a cylinder while acted upon by external pressure.

C. THE STRESSES IN THE CYLINDER DURING EXTERNAL PRESSURE APPLICATION AFTER PLASTIC YIELDING HAS OCCURRED THROUGHOUT THE CYLINDER

The pressure required to plastically yield the entire cylinder to completely autofrettage is obtained by letting n = b in equation (18). Then this pressure denoted as p_{max} is



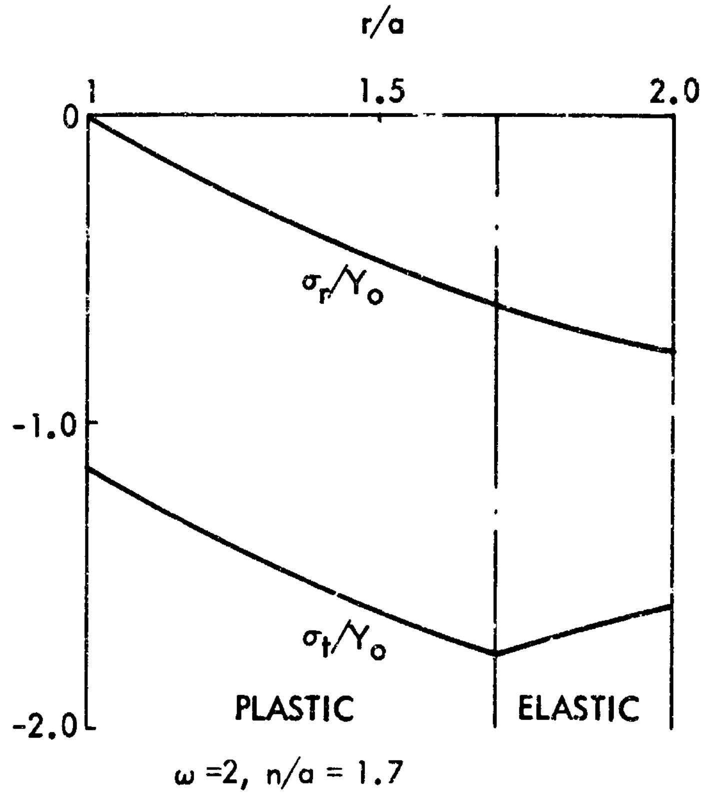


Fig. 1. External Pressure - Partial Yielding

$$p \text{ to entirely yield cylinder} = p_{max} = \frac{2Yo}{\sqrt{3}} \ln w \qquad (21)$$

where w is the wall ratio defined as

$$w = \frac{b}{a} \qquad (22)$$

The stresses existing in the completely yielded cylinder are given by equations (12) and (13) for the plastic zone which are now applicable to the entire cylinder.

$$\sigma_r = -\frac{2Yo}{\sqrt{3}} \ln \frac{r}{a} \qquad b \geq r \geq a$$

$$\sigma_t = -\frac{2Yo}{\sqrt{3}} \left(\ln\frac{r}{a} + 1\right) \qquad b \geq r \geq a$$

D. THE RESIDUAL STRESSES IN A CYLINDER WHEN REVERSE YIELDING DOES NOT OCCUR

Let us examine what occurs when the external pressure is decreased to zero. Because the cylinder had been partially or entirely plastically yielded, after the pressure is released, there remain residual stresses in the cylinder. The situation where the residual stresses are not sufficient to cause reverse yielding is considered; thus, upon release of the external pressure only elastic deformations occur. Since only elastic deformations occur, the superposition principle may be used, viz.,

$$\sigma^*_{\text{stress after pressure release (residual stress)}} = \sigma_{\text{before pressure release}} + \sigma_{\text{due to decrease of external pressure to zero}} \tag{23}$$

The asterisk here denotes the residual stress. The second term on the right of equation (23) may be evaluated from equations (1) and (2) for elastic stresses.

$$\sigma_{r\ \text{due to decrease of external pressure to zero}} = \frac{p}{w^2-1}\left|w^2 - \left(\frac{b}{r}\right)^2\right| \tag{24}$$

$$\sigma_{t\ \text{due to decrease of external pressure to zero}} = \frac{p}{w^2-1}\left|w^2 + \left(\frac{b}{r}\right)^2\right| \tag{25}$$

With the above equations the residual stresses in the previously yielded region I become

$$\sigma_r{}^* = -\frac{2Yo}{\sqrt{3}} \ln\frac{r}{a} + \frac{p}{w^2-1}\left|w^2 - \left(\frac{b}{r}\right)^2\right|, \qquad n \geq r \geq a \text{ region I} \tag{26}$$

$$\sigma_t{}^* = -\frac{2Yo}{\sqrt{3}} \left(\ln\frac{r}{a} + 1\right) + \frac{p}{w^2-1}\left|w^2 + \left(\frac{b}{r}\right)^2\right| \qquad n \geq r \geq a \text{ region I} \tag{27}$$

With the expression for p (equation (18))

$$p = \frac{Yo}{\sqrt{3}}\left|2\ln\frac{n}{a} + 1 - \left(\frac{n}{b}\right)^2\right| \tag{18}$$

and the above equations, the expression for the difference $\sigma_t^* - \sigma_r^*$ becomes

$$\sigma_t^* - \sigma_r^* = -\frac{2Yo}{\sqrt{3}} + \frac{2p}{w^2-1}\left(\frac{b}{r}\right)^2 = -\frac{2Yo}{\sqrt{3}}\Big\{1 -$$

$$\frac{\left(\frac{b}{r}\right)^2 \left|1 + 2\ln\frac{n}{a}\right| - \left(\frac{n}{r}\right)^2}{w^2-1} \qquad \begin{array}{l} n \geq r \geq a \\ \text{region I} \end{array} \qquad (28)$$

In the previously elastically deformed region II the residual stresses become from equations (24), (25), (19), and (20)

$$\sigma_r^* = -\frac{Yo}{\sqrt{3}}\left|2\ln\frac{n}{a} + 1 - \left(\frac{n}{r}\right)^2\right| + \frac{p}{w^2-1}\left|w^2 - \left(\frac{b}{r}\right)^2\right| \qquad (29)$$

$$\sigma_t^* = -\frac{Yo}{\sqrt{3}}\left|2\ln\frac{n}{a} + 1 + \left(\frac{n}{r}\right)^2\right| + \frac{p}{w^2-1}\left|w^2 + \left(\frac{b}{r}\right)^2\right| \qquad (30)$$

which became with the equation (18) for p inserted

$$\sigma_r^* = \frac{Yo}{\sqrt{3}}\,\frac{\left(2\ln\frac{n}{a} + 1\right)\left|1 - \left(\frac{b}{r}\right)^2\right| + w^2\left(\frac{n}{r}\right)^2 - \left(\frac{n}{a}\right)^2}{w^2-1} \qquad \begin{array}{l} b \geq r \geq n \\ \text{region II} \end{array} \qquad (31)$$

$$\sigma_t^* = \frac{Yo}{\sqrt{3}}\,\frac{\left(2\ln\frac{n}{a} + 1\right)\left|1 + \left(\frac{b}{r}\right)^2\right| - w^2\left(\frac{n}{r}\right)^2 - \left(\frac{n}{a}\right)^2}{w^2-1} \qquad \begin{array}{l} b \geq r \geq n \\ \text{region II} \end{array} \qquad (32)$$

Also

$$\sigma_t{}^* - \sigma_r{}^* = -\frac{2Yo}{\sqrt{3}}\left(\frac{n}{r}\right)^2 + \frac{2p}{w^2-1}\left(\frac{b}{r}\right)^2 =$$

$$\frac{2Yo}{\sqrt{3}} \frac{\left|2 \ln \frac{n}{a} + 1\right| \left(\frac{b}{r}\right)^2 - w^2 \left(\frac{n}{r}\right)^2}{w^2-1} \qquad (33)$$

$b \geq r \geq n$
region II

The magnitude and sign of these residual stresses are obtained by inserting values of radius r into the above equations. Thus, at the outside of the cylinder $r = b$, the combined stress from equation (33) is

$$S^* = \frac{\sqrt{3}(\sigma_t{}^* - \sigma_r{}^*)}{2} = \frac{Yo}{w^2-1}\left|2 \ln \frac{n}{a} + 1 - \left(\frac{n}{a}\right)^2\right| \qquad (34)$$

at $r = b$

This stress is seen to be always compressive (negative). A few values are tabulated below.

Residual Stress at $r = b$

$\frac{n}{a}$ →	1.1	1.5	2	2.2
$\frac{S^*(w^2-1)}{Yo}$ →	-.019	-.44	-1.613	-2.26

At the interface $r = n$, the combined stress is

$$S^* = \frac{\sqrt{3}}{2}(\sigma_t{}^* - \sigma_r{}^*) = \frac{Yo\left(\frac{b}{n}\right)^2}{w^2-1}\left|2 \ln \frac{n}{a} + 1 - \left(\frac{n}{a}\right)^2\right| \qquad (35)$$

at $r = n$.

The residual stress at the interface is also always compressive; in fact, it is seen that

$$\frac{S^*_{r=b}}{S^*_{r=n}} = \left(\frac{b}{n}\right)^2 \qquad (36)$$

At the inner bore $r = a$ the combined residual stress may be obtained from equation (28) as

$$S^* = \frac{\sqrt{3}}{2}\left(\sigma_t{}^* - \sigma_r{}^*\right) = \frac{Y_o}{w^2-1}\left[1 + 2w^2 \ln\frac{n}{a} - \left(\frac{n}{a}\right)^2\right] \qquad (37)$$

$$\text{at } r = a$$

This stress is seen to be always positive or tensile.

The radius at which the combined residual stress is zero is found from equation (28) to be

$$\left(\frac{r}{b}\right)^2 = \frac{1 + 2\ln\frac{n}{a} - \left(\frac{n}{b}\right)^2}{w^2-1} \qquad (38)$$

at which position $S^* = 0$. It is thus seen from the relatively small value of r that most of the cylinder is left in a state of residual compression.

A plot of the combined residual stresses in a particular case is given in figure 2.

If the cylinder has been initially entirely yielded by the external pressure, the residual stresses are given by equations (26), (27), and (28) with n equal to b. Thus,

$$\sigma_r{}^* = -\frac{2Y_o}{\sqrt{3}}\ln\frac{r}{a} + \frac{p}{w^2-1}\left|w^2 - \left(\frac{b}{r}\right)^2\right| \qquad (39)$$

$$\sigma_t{}^* = -\frac{2Y_o}{\sqrt{3}}\left(\ln\frac{r}{a} + 1\right) + \frac{p}{w^2-1}\left|w^2 + \left(\frac{b}{r}\right)^2\right| \qquad (40)$$

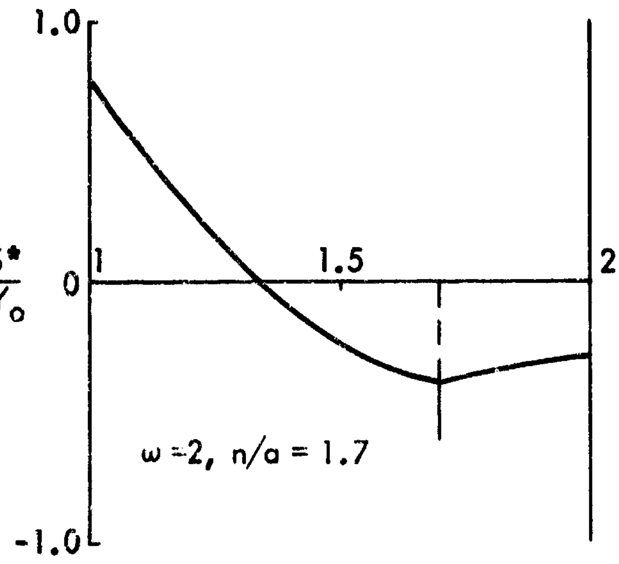


Fig. 2. Combined Residual Stress Distribution in Partially Yielded Cylinder After External Pressure is Released

$$S^* = \frac{\sqrt{3}}{2} (\sigma_t^* - \sigma_r^*) = Yo \left| \frac{2 \ln w}{w^2-1} \left(\frac{b}{r}\right)^2 - 1 \right| \tag{41}$$

where

$$p = \frac{2Yo}{\sqrt{3}} \ln w \tag{42}$$

A plot of the residual stresses on a cylinder which had been completely yielded is shown in figure 3.

E. CRITERIA FOR REVERSE YIELDING TO OCCUR

If the tensile residual stress at the bore becomes sufficiently large, "reverse yielding" in tension will occur. This situation begins when the combined residual stress at the inner bore becomes equal to +Yo, or

$$S^*_{r = a} = \frac{\sqrt{3}}{2} (\sigma_t^* - \sigma_r^*) = + Yo \tag{43}$$

(Here the yield stress in tension is taken equal in absolute value to that in compression.)

For the completely initially yielded cylinder the reverse yielding condition is obtained by substituting the yield condition equation (43) into equation (41). Thus,

$$Yo = Yo \left| \frac{2 \ln w}{w^2 - 1} (w^2) - 1 \right|$$

or

$$\ln w = \frac{w^2 - 1}{w^2} \tag{44}$$

or

$$w = 2.22 \tag{45}$$

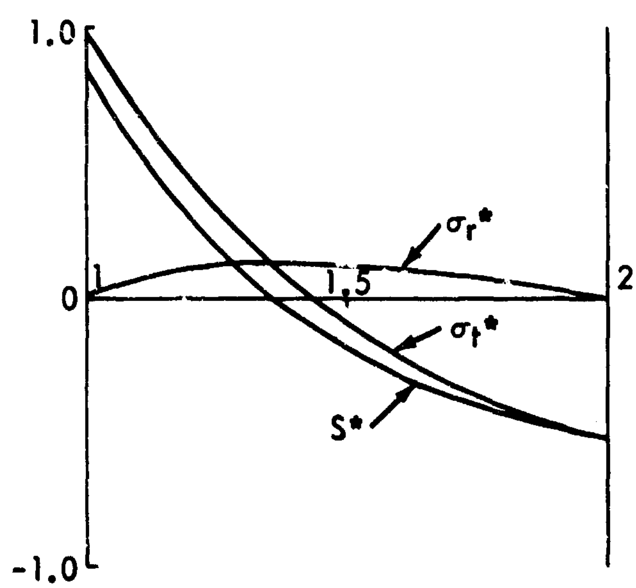


Fig. 3. Residual Stresses in a Completely Yielded Cylinder

for the bore to begin to reverse yield in a completely yielded cylinder. The corresponding pressure required to initiate this reverse yielding is from (42)

$$p = \frac{2Yo}{\sqrt{3}} \frac{w^2-1}{w^2} = \frac{1.59\ Yo}{\sqrt{3}} = .92\ Yo \tag{46}$$

In the case of a cylinder which is not initially completely yielded, reverse yielding may be found from equation (37).

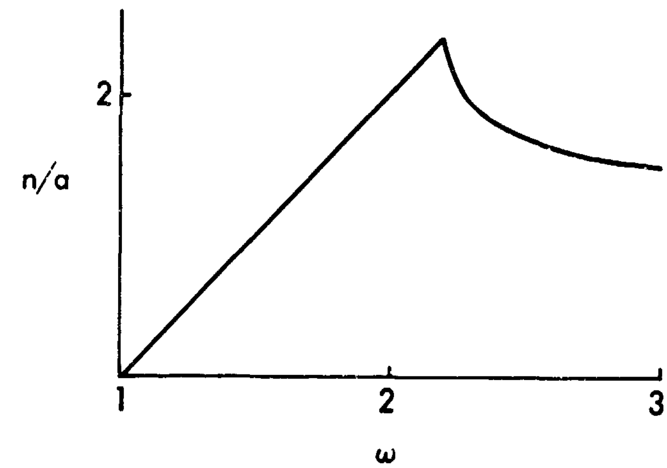


Fig. 4(a). n/a vs w to Limit the Residual Tension of Bore to the Yield Point of Material

$$S^* = Yo = \frac{Yo}{w^2-1} \left| 1 + 2w^2 \ln \frac{n}{a} - \left(\frac{n}{a}\right)^2 \right| \tag{47}$$

from which

$$w^2 = \frac{\left(\frac{n}{a}\right)^2 - 2}{2 \ln \frac{n}{a} - 1} \tag{48}$$

to initiate reverse yielding. This relation is plotted in figure 4(a). The pressure required to initiate reverse yielding may be found by substituting equation (48) into the pressure equation (18). Then it is found that

$$p_{\text{to initiate reverse yielding}} = \frac{2Yo}{\sqrt{3}} \frac{w^2-1}{w^2} \tag{49}$$

A few values of $\frac{n}{a}$, and p for given w values are tabulated below.

Initiation of Reverse Yielding

w	2.2	2.27	2.37	2.66	2.97	3.82	∞
$\frac{n}{a}$	2.2	2.0	1.9	1.8	1.75	1.70	1.65
$\frac{p}{Yo}$	.919	.931	.949	.991	1.024	1.078	1.153

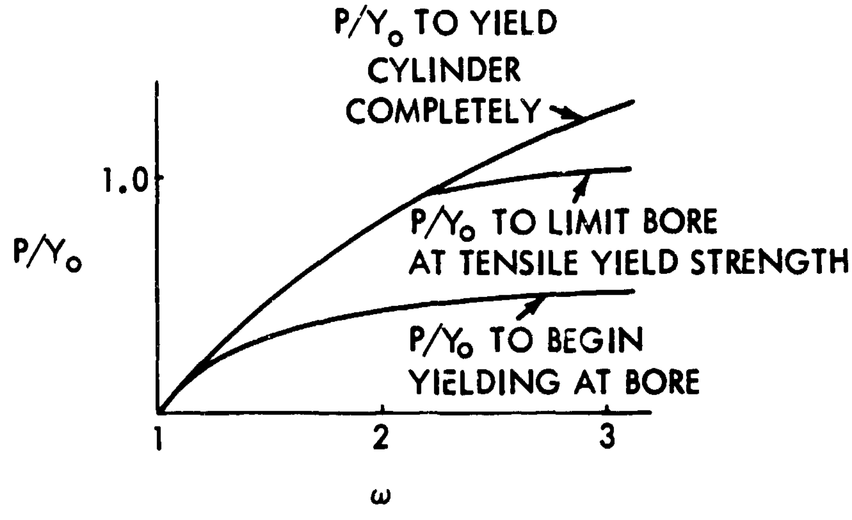


Fig. 4(b). p/Yo vs w for Various Yielding Conditions

In figure 4(b) is plotted the pressure to initiate reverse yielding as a function of wall ratio. Also plotted in this figure are the values of external pressure to begin yielding and to completely yield the cylinder.

F. REMARKS CONCERNING SIMILARITY OF CYLINDER BEHAVIOR RESULTING FROM EXTERNAL PRESSURE TO THAT FROM INTERNAL PRESSURE

It is of interest to compare the results obtained here for the behavior of a cylinder resulting from external pressure application to those for a cylinder having internal pressure application. The internal pressure case is discussed in reference 2. By comparing the results of reference 2 with the present results, the following is noted:

The individual stresses in the two cases are unrelated; however, the combined stress S, during pressure application and the residual combined stress S* after pressure release and reapplication are identically the same in magnitude at every point in the cylinder but of opposite sign.

Thus, the combined stress is the same function of applied pressure and wall ratio at every point in the cylinder for the two cases. Yielding to given radii will occur at the same values of pressure. (Of course, as the sign is different, the yielding will be compressive in one case, and tensile in the other.) Reapplication of pressure will cause yielding to the same radii.

Similarly, for a given cylinder, reyielding will be initiated at the release of the same value of pressure for the two cases.

G. THE REVERSE YIELDING SITUATION FOR THE ENTIRELY PLASTICALLY YIELDED CYLINDER

As shown above for a cylinder which has been entirely yielded by external pressure with wall ratios w greater than 2.22, upon release of the pressure, reverse yielding will occur. The behavior of the entirely plastically yielded cylinder will be here obtained. The method used will be similar to that of reference 4. The assumptions made are the following:

1. The material is perfectly plastic

2. $\sigma_z = \frac{1}{2}(\sigma_t + \sigma_r)$

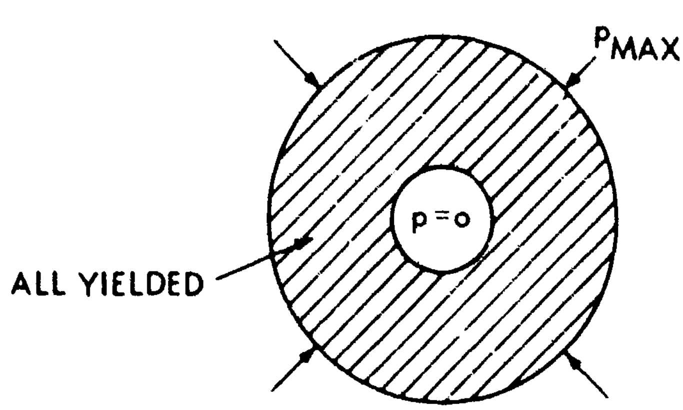
3. The yield criterion is given by the Distortion Energy Theory.

Assumptions (2) and (3) result in the following yield criterion:

$$\sigma_t - \sigma_r = \pm \frac{2Yo}{\sqrt{3}}$$

Let us consider the case of the fully yielded cylinder of w 2.2 under external pressure p_{max}. As shown above

$$\frac{p_{max}}{Yo} = \frac{2}{\sqrt{3}} \ln w \tag{50}$$



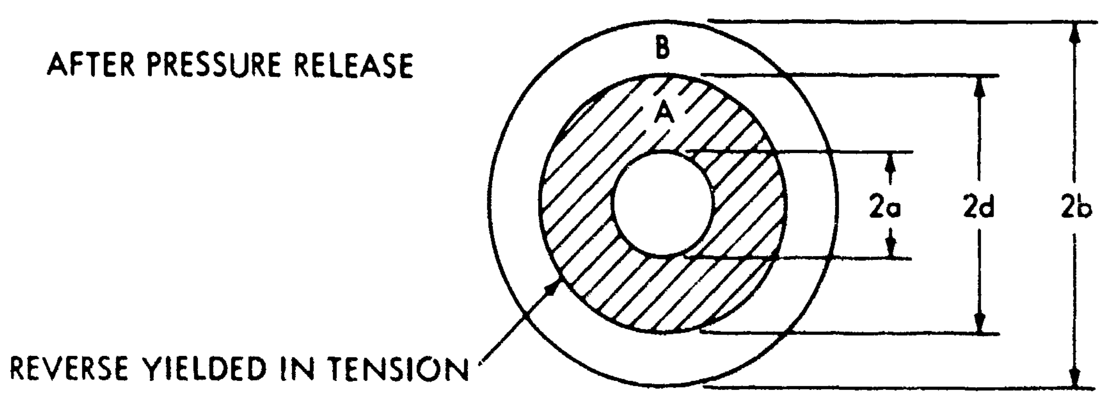
and the stresses existing are

$$\sigma_r = - \frac{2Yo}{\sqrt{3}} \ln \frac{r}{a} \tag{51}$$

and

$$\sigma_t = - \frac{2Yo}{\sqrt{3}} \left(\ln \frac{r}{a} + 1\right) \qquad b \geqslant r \geqslant a \tag{52}$$

As the external pressure is released, the cylinder deforms elastically until the bore reaches the yield point in tension. Thereafter, as the pressure is further reduced, plastic flow progresses outward from the bore. When the external pressure reaches zero, the cylinder will consist of two zones, an inner core which has reverse yielded in tension and an outer elastic jacket that has been previously yielded in compression during application of the external pressure.



The applicable equations for the reverse yielded core after pressure release are the yield criterion in tension and the equilibrium equation, viz.

$$\sigma_t^* - \sigma_r^* = - \frac{2Yo}{\sqrt{3}} \qquad d \geqslant r \geqslant a \tag{53}$$

$$\sigma_t^* - \sigma_r^* - r \frac{d\sigma_r^*}{dr} = 0 \tag{54}$$

These equations with the fact that the residual radial stress at the bore is zero lead to the following equations for the

inner core stresses after pressure release.

$$\sigma_r^* = \frac{2Yo}{\sqrt{3}} (\ln \frac{r}{a}) \qquad a \leqslant r \leqslant d \text{ (zone A)} \qquad (55)$$

$$\sigma_t^* = \frac{2Yo}{\sqrt{3}} (\ln \frac{r}{a} + 1) \qquad a \leqslant r \leqslant d \text{ (zone A)} \qquad (56)$$

where d denotes the radius at the interface. These are the residual tensile stresses after pressure release in the reverse yielded inner core.

Since during pressure release the outer jacket is only deformed elastically, the stresses may be obtained by superposition of the elastic stresses (Lamé) on the existing stresses before pressure release. The elastic stresses are of the form (this form follows from Hooke's law and the equilibrium condition):

$$\sigma_{r \text{ elastic}} = E - \frac{F}{r^2} \qquad (57)$$

$$\sigma_{t \text{ elastic}} = E + \frac{F}{r^2} \qquad (58)$$

The stresses existing before pressure release are given by equations (51) and (52). Hence, the residual stresses after pressure release in the outer jacket are the sum of the above, viz.

$$\sigma_r^* = E - \frac{F}{r^2} - \frac{2Yo}{\sqrt{3}} \ln \frac{r}{a} \qquad b \geqslant r \geqslant d \qquad (59)$$

$$\sigma_t^* = E + \frac{F}{r^2} - \frac{2Yo}{\sqrt{3}} (\ln \frac{r}{a} + 1) \qquad (60)$$

where E and F are constants to be determined.

At the interface r = d, the above stresses are equal to those in the inner core. Hence,

$$E - \frac{F}{d^2} - \frac{2Yo}{\sqrt{3}} \ln \frac{d}{a} = \frac{2Yo}{\sqrt{3}} \ln \frac{d}{a} \tag{61}$$

$$E + \frac{F}{d^2} - \frac{2Yo}{3} \left(\ln \frac{d}{a} + 1\right) = \frac{2Yo}{\sqrt{3}} \left(\ln \frac{d}{a} + 1\right) \tag{62}$$

From the above equations

$$E = \frac{2Yo}{\sqrt{3}} \left(1 + 2 \ln \frac{d}{a}\right) \tag{63}$$

$$F = d^2 \frac{2Yo}{\sqrt{3}} \tag{64}$$

Therefore, the residual stresses in the outer jacket are

$$\sigma_r^* = \frac{2Yo}{\sqrt{3}} \left[1 + 2 \ln \frac{d}{a} - \left(\frac{d}{r}\right)^2 - \ln \frac{r}{a}\right] \tag{65}$$

$$\sigma_t^* = \frac{2Yo}{\sqrt{3}} \left[2 \ln \frac{d}{a} + \left(\frac{d}{r}\right)^2 - \ln \frac{r}{a}\right] \quad \begin{matrix} b \geq r \geq d \\ \text{(zone B)} \end{matrix} \tag{66}$$

The combined residual stress is

$$S^* = \frac{\sqrt{3}}{2} (\sigma_t^* - \sigma_r^*) = Yo \left[2 \left(\frac{d}{r}\right)^2 - 1\right] \tag{67}$$

Since at the outer radius $r = b$, the residual radial stress σ_r^* is zero, from equation (65) above

$$2 \ln \frac{d}{a} + 1 = \left(\frac{d}{b}\right)^2 + \ln w \tag{68}$$

which may be rewritten as

$$\ln \frac{d}{a} = \left(\frac{d}{b}\right)^2 + \ln \frac{b}{d} - 1 \tag{69}$$

The extent of the reversed yielding can be calculated from this equation by solving for the inner core radius d.

Figure 5 shows the plastic and elastic zone dimensions of a cylinder of wall ratio, w equal to 5. The value of $\frac{d}{a}$ is calculated from equation (69) to be 1.4. It is seen that the plastic inner core is relatively small. Figure 6 is a plot of $\frac{d}{a}$ for various wall ratios.

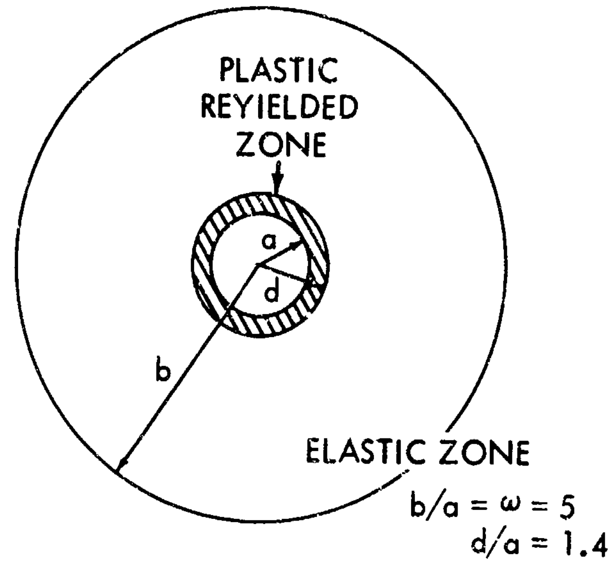


Fig. 5. Comparative Plastic and Elastic Zones in a Reyielded Cylinder

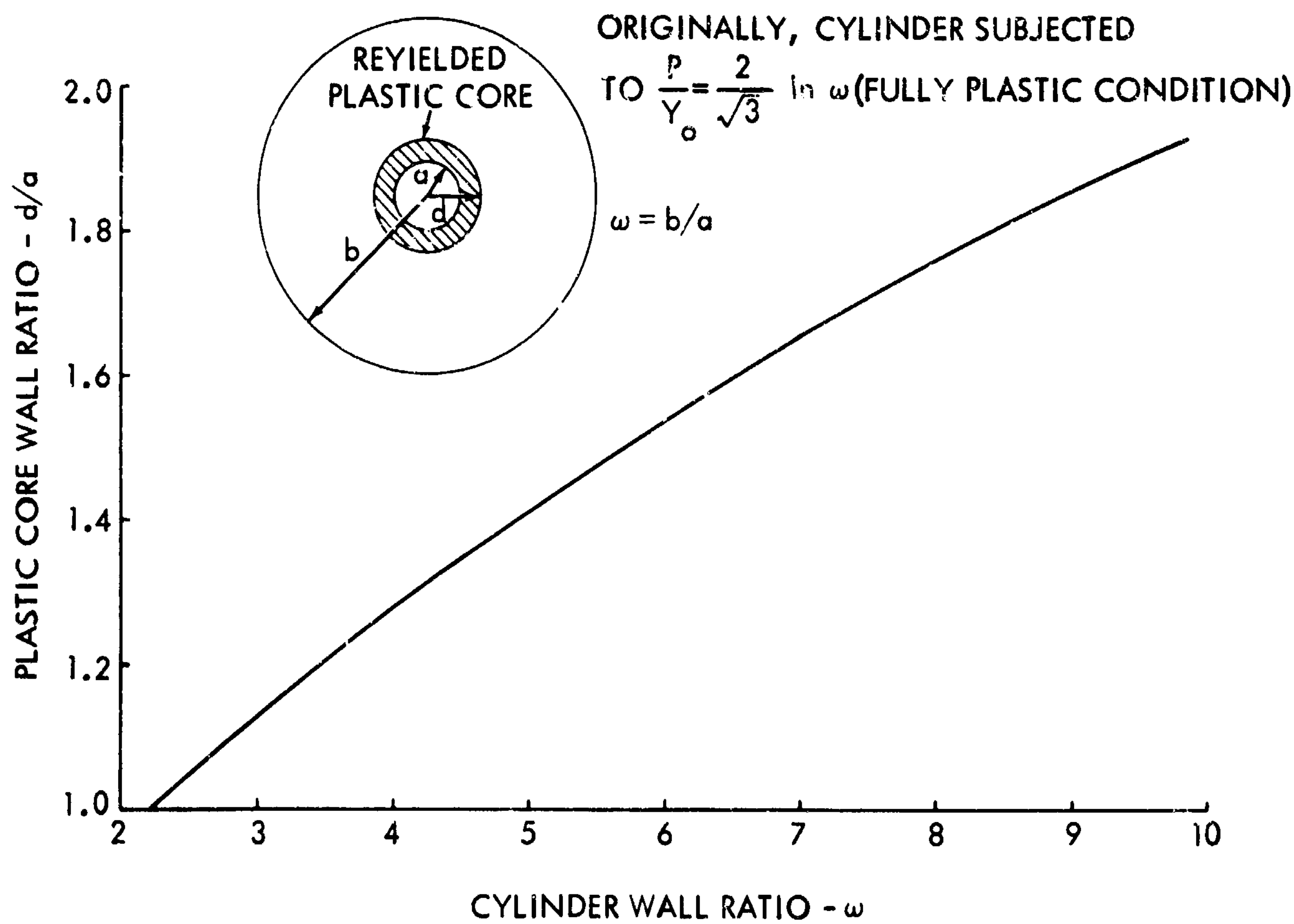


Fig. 6. Wall Ratio of Reyielded Plastic Core vs Wall Ratio of Cylinder

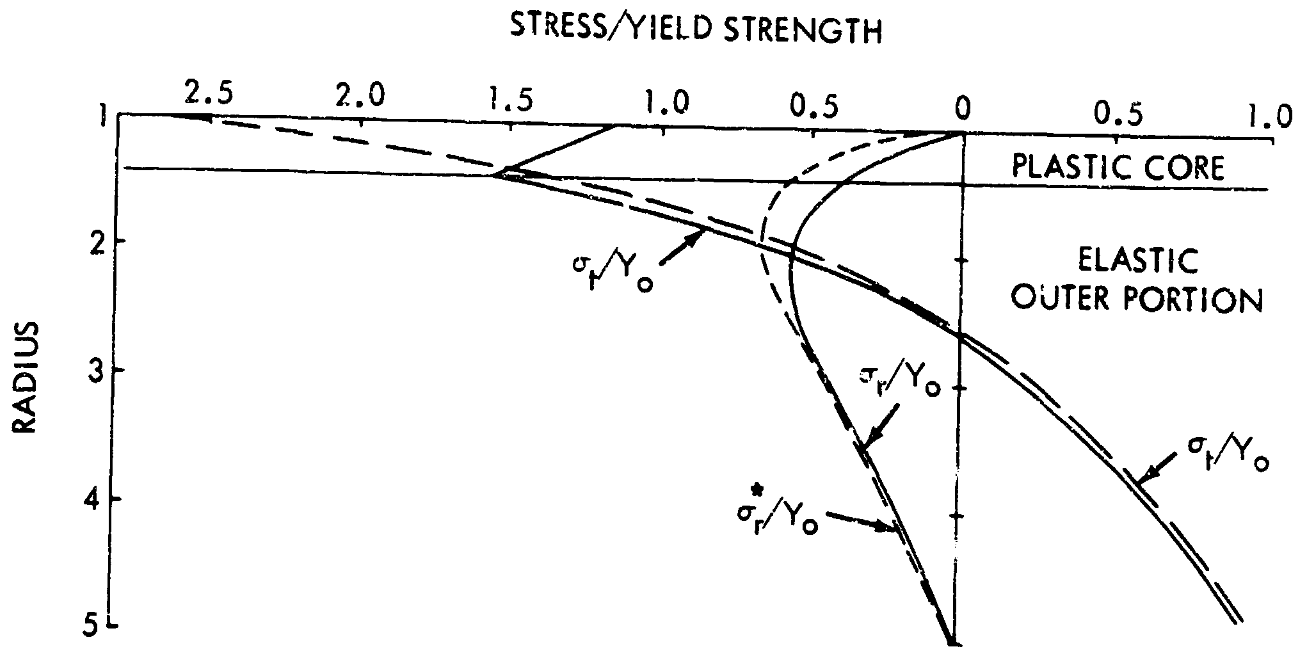


Fig. 7. Stress Distribution in Reyielded Cylinder of Wall Ratio 5

Figure 7 shows the residual stress distributions in a reverse yielded cylinder of w = 5. Included in the plot, as dotted lines, are the residual stresses σ_r* and σ_t* that would exist if the cylinder had no limiting tensile yield strength. It can be seen that these residual stresses are only slightly modified in the elastic zone.

The results indicate that if external pressure is applied to a cylinder with wall ratio greater than 2.22 so as to yield it completely in compression, upon release of this pressure there will be a relatively small inner core of material which is reverse yielded in tension.

H. PRESSURE APPLICATION TO THE REVERSE YIELDED CYLINDER

If external pressure is reapplied to the reverse yielded cylinder, initially the deformations will be elastic; the stresses then existing are the residual stresses plus elastic Lamé stresses resulting from the external pressure

$$\sigma_r = \sigma_r^* + \text{Lame}' \text{ stress} = \sigma_r^* - \frac{p}{w^2-1}\left[w^2 - \left(\frac{b}{r}\right)^2\right] \qquad (70)$$

$$\sigma_t = \sigma_t^* + \text{Lame}' \text{ stress} = \sigma_t^* - \frac{p}{w^2-1}\left[w^2 + \left(\frac{b}{r}\right)^2\right] \qquad (71)$$

The residual stresses are those in equations (55), (56), (59), and (60). Inserting these into the above, one obtains

$$\sigma_r = \frac{2Yo}{\sqrt{3}}\left[1+2\ln\frac{d}{a} - \left(\frac{d}{r}\right)^2 - \ln\frac{r}{a}\right] - \frac{p}{w^2-1}\left[w^2 - \left(\frac{b}{r}\right)^2\right] \qquad (72)$$

$$b \geq r \geq d$$

$$\sigma_t = \frac{2Yo}{\sqrt{3}}\left[2\ln\frac{d}{a} + \left(\frac{d}{r}\right)^2 - \ln\frac{r}{a}\right] - \frac{p}{w^2-1}\left[w^2 + \left(\frac{b}{r}\right)^2\right] \qquad (73)$$

$$b \geq r \geq d$$

and

$$\sigma_r = \frac{2Yo}{\sqrt{3}}\ln\frac{r}{a} - \frac{p}{w^2-1}\left[w^2 - \left(\frac{b}{r}\right)^2\right] \qquad a \leq r \leq d \qquad (74)$$

$$\sigma_t = \frac{2Yo}{\sqrt{3}}\left(\ln\frac{r}{a} + 1\right) - \frac{p}{w^2-1}\left[w^2 + \left(\frac{b}{r}\right)^2\right] \qquad a \leq r \leq d \qquad (75)$$

$$S = \frac{\sqrt{3}}{2}(\sigma_t - \sigma_r) = Yo - \frac{p}{w^2-1}\sqrt{3}\left(\frac{b}{r}\right)^2 \qquad a \leq r \leq d \qquad (76)$$

As the pressure is increased, the combined stress S at the bore is increased; the external pressure necessary to initiate compressive yielding at the bore is found from equation (76) above by letting S = -Yo and r = a;

$$S = -Yo = Yo - \frac{p}{w^2-1}\sqrt{3}\;w^2 \qquad (77)$$

or

$$p = + \frac{2Yo}{\sqrt{3}} \frac{w^2-1}{w^2} \tag{78}$$

Thus, the value of the pressure necessary to cause the bore to yield the third time is identically the same as that value required when released to initiate reverse yielding (see equation (46)).

If the external pressure is increased beyond the value of equation (78) above, the inner bore will begin to yield in compression. Further application of the external pressure will cause the region of plastic deformation to extend radially from the bore to, say, a radius c. For this plastic region the yield criterion

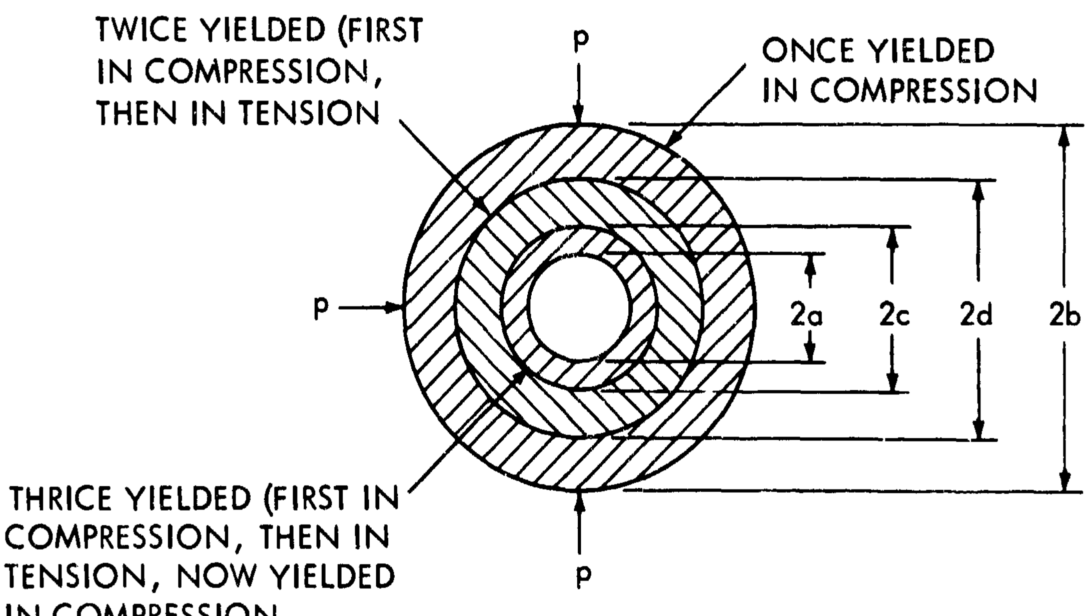
$$\sigma_t - \sigma_r = - \frac{2Yo}{\sqrt{3}} \tag{79}$$

the equilibrium equation (11), and the boundary condition that the radial stress at the inside surface is equal to zero result in the following

$$\sigma_r = - \frac{2Yo}{\sqrt{3}} \ln \frac{r}{a} \qquad c \geq r \geq a \tag{80}$$

$$\sigma_t = - \frac{2Yo}{\sqrt{3}} \left(1 + \ln \frac{r}{a}\right) \qquad c \geq r \geq a \tag{81}$$

These are the stresses in the thrice yielded core of the cylinder. The cylinder at this time appears as sketched below.



Because the deformations in the cylinder other than in the thrice yielded core are elastic, the principle of superposition may be again used. The stresses are thus the sum of existing residual stresses before pressure reapplications plus the Lamé elastic stresses.

$$\sigma_r = G - \frac{H}{r^2} + \sigma_r^*$$

$$= G - \frac{H}{r^2} + \frac{2Yo}{\sqrt{3}} \ln \frac{r}{a} \qquad d \geq r \geq c \qquad (82)$$

$$\sigma_t = G + \frac{H}{r^2} + \sigma_t^*$$

$$= G + \frac{H}{r^2} + \frac{2Yo}{\sqrt{3}} \left(\ln \frac{r}{a} + 1\right) \qquad d \geq r \geq c \qquad (83)$$

At $r = c$, the stresses of the core equations (80) and (81) are equal to the above stresses.

$$-\frac{2Yo}{\sqrt{3}} \ln \frac{c}{a} = G - \frac{H}{c^2} + \frac{2Yo}{\sqrt{3}} \ln \frac{c}{a}$$

$$-\frac{2Yo}{\sqrt{3}} \left(1 + \ln \frac{c}{a}\right) = G + \frac{H}{c^2} + \frac{2Yo}{\sqrt{3}} \left(\ln \frac{c}{a} + 1\right)$$

From which

$$G = -\frac{2Yo}{\sqrt{3}} \left(1 + 2 \ln \frac{c}{a}\right) \qquad (84)$$

$$H = -\frac{2Yo}{\sqrt{3}} (c^2) \qquad (85)$$

and the stresses are thus

$$\sigma_r = - \frac{2Yo}{\sqrt{3}} \left[1 + 2 \ln \frac{c}{a} - (\frac{c}{r})^2 - \ln \frac{r}{a}\right] \qquad d \geq r \geq c \qquad (86)$$

$$\sigma_t = - \frac{2Yo}{\sqrt{3}} \left[2 \ln \frac{c}{a} + (\frac{c}{r})^2 - \ln \frac{r}{a}\right] \qquad d \geq r \geq c \qquad (87)$$

and

$$(\sigma_t - \sigma_r) \frac{\sqrt{3}}{2} = S = Yo \left[1 - 2(\frac{c}{r})^2\right] \qquad d \geq r \geq c \qquad (88)$$

In the region $b \geq r \geq d$

$$\sigma_r = J - \frac{K}{r^2} + \sigma_r^*$$

$$= J - \frac{K}{r^2} + \frac{2Yo}{\sqrt{3}}\left[1 + 2 \ln \frac{d}{a} - (\frac{d}{r})^2 - \ln \frac{r}{a}\right] \qquad (89)$$

$$\sigma_t = J + \frac{K}{r^2} + \sigma_t^*$$

$$= J + \frac{K}{r^2} + \frac{2Yo}{\sqrt{3}}\left[2 \ln \frac{d}{a} + (\frac{d}{r})^2 - \ln \frac{r}{a}\right] \qquad (90)$$

From the fact that at $r = d$ the above stresses are equal to those of equations (86) and (87), the following results are obtained:

$$J = - \frac{2Yo}{\sqrt{3}} (1 + 2 \ln \frac{c}{a}) \qquad (91)$$

$$K = - \frac{2Yo}{\sqrt{3}} (c)^2 \qquad (92)$$

$$\sigma_r = \frac{2Yo}{\sqrt{3}} \left[2 \ln \frac{d}{c} + \left(\frac{c}{r}\right)^2 - \left(\frac{d}{r}\right)^2 - \ln \frac{r}{a} \right] \quad b \geq r \geq d \qquad (93)$$

$$\sigma_t = \frac{2Yo}{\sqrt{3}} \left[2 \ln \frac{d}{c} - \left(\frac{c}{r}\right)^2 + \left(\frac{d}{r}\right)^2 - \ln \frac{r}{a} - 1 \right] \quad b \geq r \geq d \qquad (94)$$

$$S = \frac{\sqrt{3}}{2}(\sigma_t - \sigma_r) = Yo \left[2 \left(\frac{d}{r}\right)^2 - 2\left(\frac{c}{r}\right)^2 - 1 \right] \quad b \geq r \geq d \qquad (95)$$

At the outermost radius ($r = b$) the radial stress is equal to the negative of the applied external pressure. Hence, from equation (93)

$$- p = \frac{2Yo}{\sqrt{3}} \left[2 \ln \frac{d}{c} + \left(\frac{c}{b}\right)^2 - \left(\frac{d}{b}\right)^2 - \ln w \right] \qquad (96)$$

which may be rewritten from equation (68) as

$$p = \frac{2Yo}{\sqrt{3}} \left[1 + 2 \ln \frac{c}{a} - \left(\frac{c}{b}\right)^2 \right] \qquad (97)$$

This expression relates the pressure p with the radius c to which plastic flow has been reached.

If the pressure is further increased, the plastic flow reaches the radius d (and $d = c$). The pressure required is obtained by inserting $c = d$ in equation (96) from which

$$+ p = + \frac{2Yo}{\sqrt{3}} \ln w \qquad (98)$$

Then the combined stress S is equal to the yield stress in compression at this radius.

$$S = -Yo \quad \text{at } c = d \tag{99}$$

At this time with c = d the stresses above equations(93), (94), and (95) are seen to become

$$\sigma_r = -\frac{2Yo}{\sqrt{3}} \ln \frac{r}{a} \qquad b \geqslant r \geqslant d \tag{100}$$

$$\sigma_t = -\frac{2Yo}{\sqrt{3}} \left(\ln \frac{r}{a} + 1\right) \qquad b \geqslant r \geqslant d \tag{101}$$

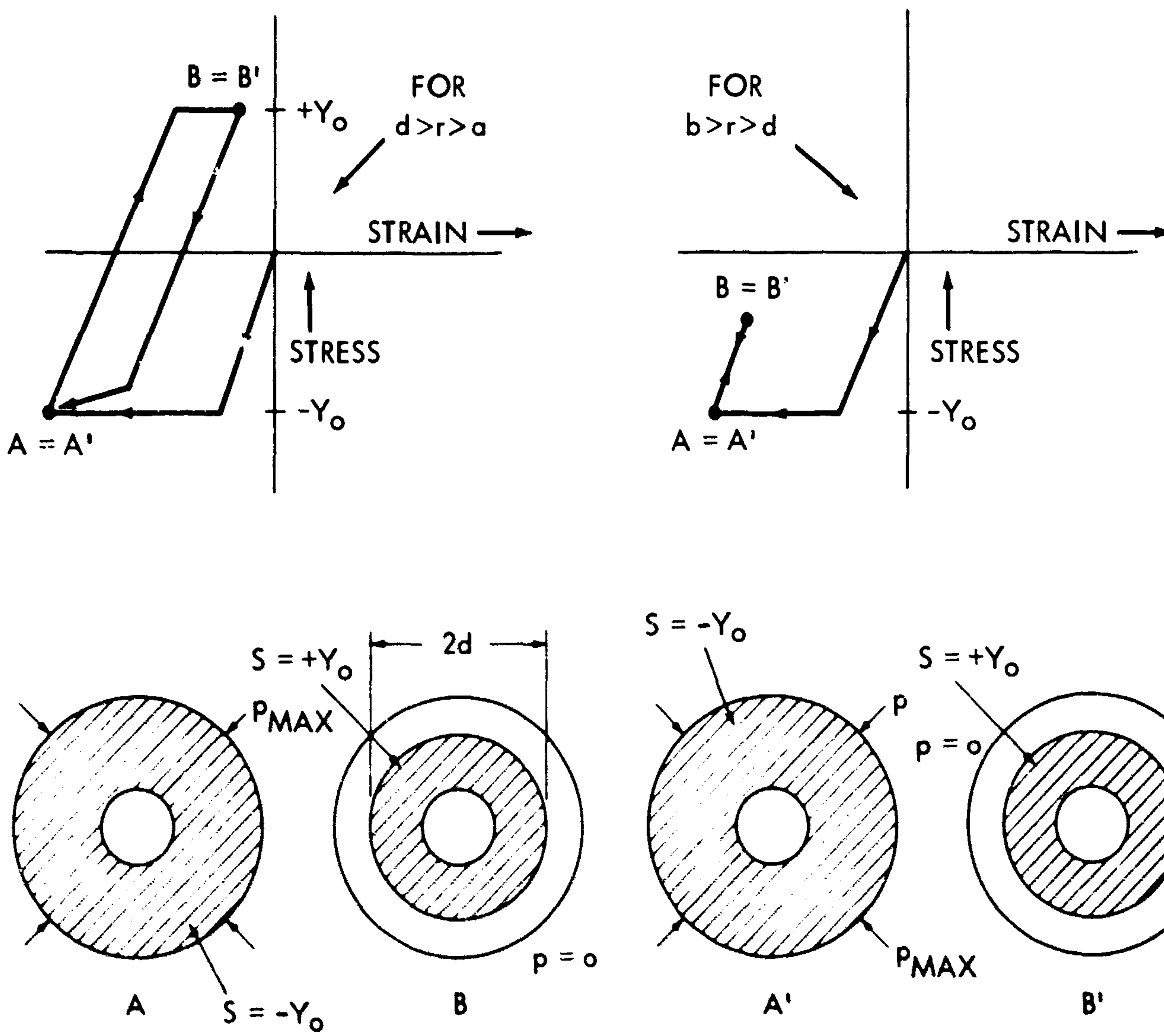
$$S = -Yo \qquad b \geqslant r \geqslant d \tag{102}$$

It is seen that the combined stress in the region $b \geqslant r \geqslant d$ is independent of radius r; the above equations indicate that this outermost region which was elastic in the reverse yielded cylinder becomes plastic instantaneously when the plastic region reaches the radius d during reapplication of the pressure. The value of pressure required to effect this sudden change which yields the entire cylinder

$$p = \frac{2Yo}{\sqrt{3}} \ln w$$

is observed to be the same as originally required to yield the entire cylinder.

The stress-strain history of elements in the cylinder wall is sketched on the next page.



I. REMARKS CONCERNING SIMILARITY OF CYLINDER BEHAVIOR RESULTING FROM EXTERNAL PRESSURE TO THAT FROM INTERNAL PRESSURE

It is of interest to compare the results obtained here for the behavior of a cylinder resulting from external pressure application to those for a cylinder having internal pressure application. The internal pressure case is discussed in references 2, 3 and 4. By comparing the results of these references with the present results, the following is noted:

The individual stresses in the two cases are unrelated; however, <u>the combined stress S, during pressure application,</u>

after pressure release, and during pressure reapplication is identically the same in magnitude at every point in the cylinder but of opposite sign--the combined stress is the same function of applied pressure and wall ratio at every point in the cylinder for the two cases. Yielding to given radii will occur at the same values of pressure. (Of course, as the sign is different, the yielding will be compressive in one case, and tensile in the other.) Reverse yielding will similarly progress to equal radii at release of equal values of pressure. Upon reapplication of pressure, again yielding will progress to the same value of radius for the same pressure value.

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The equations for plastic flow in a cylinder subjected to external pressure are derived. The case where reverse yielding occurs is also considered. The derived equations give results that are very similar to those for internal pressure application.

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